GREAT ELLIPTIC ARC DISTANCE[1]

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ABSTRACT

These notes provide a detailed derivation of series formula for (i) Meridian distance M as a function of latitude ϕ and the ellipsoid constant eccentricity-squared e^2, and ϕ and the ellipsoid constant third flattening n; (ii) Rectifying latitude μ as functions of ϕ, e^2 and ϕ, n; and (iii) latitude ϕ as functions of μ, e^2 and μ, n These series can then be used in solving the direct and inverse problems on the ellipsoid (using great elliptic arcs) and are easily obtained using the Computer Algebra System *Maxima.*

In addition, a detailed derivation of the equation for the great elliptic arc on an ellipsoid is provided as well as defining the azimuth and the vertex of a great elliptic. And to assist in the solution of the direct and inverse problems the auxiliary sphere is introduced and equations developed.

INTRODUCTION

In geodesy, the great elliptic arc between P_1 and P_2 on the ellipsoid is the curve created by intersecting the ellipsoid with the plane containing P_1, P_2 and O (the centre of the ellipsoid) and these planes are great elliptic planes or sections. Figure 1 shows P on the great elliptic arc between P_1 and P_2. θ_P is the geocentric latitude of P and λ_P is the longitude of P.

There are an infinite number of planes that cut the surface of the ellipsoid and contain the chord P_1P_2 but only one of these will contain the centre O. Two other planes are the normal section plane P_1P_2 (containing the normal at P_1) and the normal section plane P_2P_1

[1] This paper follows on from an earlier paper *The Great Elliptic Arc on an Ellipsoid* and is the result of work done for a review of a paper titled *The General Solutions for Great Ellipse on the Spheroid* by Wei-Kuo Tseng (Department of Merchant Marine, National Taiwan Ocean University). The methods and equations developed here follow Tseng (in part) and others mentioned in the References.

(containing the normal at P_2). All of these curves of intersection (including the great elliptic arc and the two normal section curves) are plane curves that are arcs of ellipses (for a proof of this see Deakin & Hunter, 2010a).

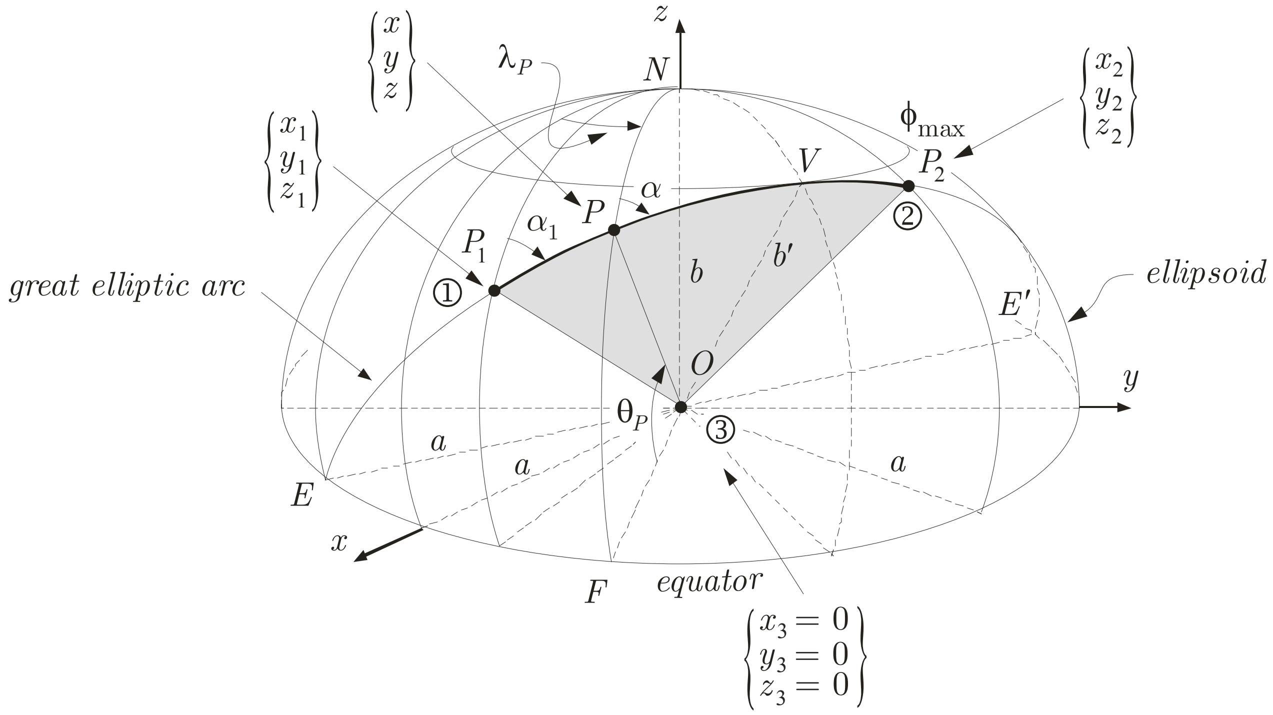

Figure 1: Great elliptic arc on ellipsoid

All meridians of longitude on an ellipsoid and the ellipsoid equator are great elliptic arcs. Parallels of latitude – excepting the equator – are not great elliptic arcs. So we could say that the great elliptic arc is a unique plane curve on the ellipsoid – since it is created by the single plane containing P_1, P_2 and O. But it is not the shortest distance between P_1 and P_2; this unique property (shortest length) belongs to the geodesic.

Great elliptic arcs are seldom used in geodesy as they don't have a practical connection with theodolite observations made on the surface of the earth that are approximated as observations made on an ellipsoid; e.g., normal section curves and curves of alignment. Nor are they the shortest distance between points on the ellipsoid; but, if we ignore earth rotation, they are the curves traced out on the geocentric ellipsoid by the ground point of an earth orbiting satellite or a ballistic missile moving in an orbital plane containing the earth's centre of mass. Here geocentric means O (the centre of the ellipsoid) is coincident with the centre of mass.

The series that we will develop are based on meridian distance *M;* the distance along the arc of a meridian from the equator to the point of latitude ϕ and meridian distances are identical to distances from E on the equator to P_1 and P_2 on the great elliptic (Figure 1).

Every great elliptic section intersects the equatorial plane of the ellipsoid along a line of intersection passing the centre O, and the equator at E and E' that are 'nodes' of the great elliptic. The line $EE' = 2a$ is the major axis of the great elliptic. Every great elliptic will have two vertices V and V' (north and south of the equator) where a point moving along the great elliptic will attain a maximum latitude and the line $VV' = 2b'$ is the minor axis of the great elliptic; perpendicular to EE' and passing through O (note that V' is not shown in Figure 1 and that $OV = b'$ is the semi-minor axis of the great elliptic). The meridian passing through the vertex will be advanced 90° in longitude from the meridian passing through the node; i.e., $\lambda_V - \lambda_E = 90^\circ$ and we will show how the latitude and longitude of the vertex can be obtained from the equation of the great elliptic plane. Knowing these relationships is the key to solving the direct and inverse problems of the great elliptic arc on the ellipsoid where the direct problem is: given latitude and longitude of P_1 and the azimuth α_{12} and distance s to P_2; compute the latitude and longitude of P_2 and the reverse azimuth α_{21}. The inverse problem is: given the latitudes and longitudes of P_1 and P_2; compute the forward and reverse azimuths α_{12}, α_{21} and the great elliptic arc distance s.

Before proceeding, some definitions of terms relating to the ellipsoid (and including a definition the ellipsoid itself) will be useful

Ellipsoid

The ellipsoid is a surface of revolution created by rotating an ellipse (whose semi-axes are a and b and $a > b$) about its minor axis. It is the mathematical approximation of the earth and has geometric constants: flattening f; third flattening n; eccentricity e; second eccentricity e' and polar radius of curvature c given by

$$f = \frac{a-b}{a} \tag{1}$$

$$n = \frac{a-b}{a+b} = \frac{f}{2-f} = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} = \frac{\sqrt{1+e'^2}-1}{\sqrt{1+e'^2}+1} \tag{2}$$

$$e^2 = \frac{a^2-b^2}{a^2} = \frac{e'^2}{1+e'^2} = f\left(2-f\right) = \frac{4n}{\left(1+n\right)^2} \tag{3}$$

$$e'^2 = \frac{a^2-b^2}{b^2} = \frac{e^2}{1-e^2} = \frac{f\left(2-f\right)}{\left(1-f\right)^2} = \frac{4n}{\left(1-n\right)^2} \tag{4}$$

$$c = \frac{a^2}{b} = a\left(\frac{1+n}{1-n}\right) \tag{5}$$

The ellipsoid radii of curvature ρ (meridian plane) and ν (prime vertical plane) at a point whose latitude is ϕ are (Deakin & Hunter 2010a)

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{3/2}} = \frac{a\left(1-e^2\right)}{W^3} = \frac{c}{V^3} \quad \text{and} \quad \nu = \frac{a}{\left(1-e^2\sin^2\phi\right)^{1/2}} = \frac{a}{W} = \frac{c}{V} \tag{6}$$

where latitude functions V and W are

$$W^2 = 1 - e^2\sin^2\phi; \;\; V^2 = 1 + e'^2\cos^2\phi = \frac{1+n^2+2n\cos 2\phi}{\left(1-n\right)^2} \quad \text{and} \quad W = \frac{b}{a}V \tag{7}$$

Meridian distance M

Meridian distance M is defined as the arc of the meridian ellipse from the equator to the point of latitude ϕ (in Figure 1 the meridian distance of P is the arc FP)

$$M = \int_0^\phi \rho\, d\phi = \int_0^\phi \frac{a\left(1-\varepsilon^2\right)}{W^3}\, d\phi = \int_0^\phi \frac{c}{V^3}\, d\phi \tag{8}$$

This is an elliptic integral that cannot be expressed in terms of elementary functions; instead, the integrand is expanded by into a series using Taylor's theorem (see Appendix 2) then evaluated by term-by-term integration. The usual form of the series formula for M is a function of ϕ and powers of e^2 obtained from (Deakin & Hunter 2010a)

$$M = a\left(1-e^2\right)\int_0^\phi \left(1-e^2\sin^2\phi\right)^{-3/2} d\phi \tag{9}$$

But the German geodesist F.R. Helmert (1880) gave a formula for meridian distance as a function of ϕ and powers of n that required fewer terms for the same accuracy. Helmert's method of development is given in Deakin & Hunter (2010a) and with some algebra we may write

$$M = \frac{a}{1+n}\int_0^\phi \left(1-n^2\right)^2\left(1+n^2+2n\cos 2\phi\right)^{-3/2} d\phi \tag{10}$$

We will show, using Maxima, that (9) and (10) can easily be evaluated and M written as

$$M = a\left(1-e^2\right)\left\{b_0\phi + b_2\sin 2\phi + b_4\sin 4\phi + b_6\sin 6\phi + b_8\sin 8\phi + b_{10}\sin 10\phi + \cdots\right\} \tag{11}$$

where the coefficients $\left\{b_n\right\}$ are to order e^{10} as follows

$$
\begin{aligned}
b_0 &= 1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \frac{175}{256}e^6 + \frac{11025}{16384}e^8 + \frac{43659}{65536}e^{10} + \cdots \\
b_2 &= -\frac{3}{8}e^2 - \frac{15}{32}e^4 - \frac{525}{1024}e^6 - \frac{2205}{4096}e^8 - \frac{72765}{131072}e^{10} - \cdots \\
b_4 &= \frac{15}{256}e^4 + \frac{105}{1024}e^6 + \frac{2205}{16384}e^8 + \frac{10395}{65536}e^{10} + \cdots \\
b_6 &= -\frac{35}{3072}e^6 - \frac{105}{4096}e^8 - \frac{10395}{262144}e^{10} - \cdots \\
b_8 &= \frac{315}{131072}e^8 + \frac{3465}{524288}e^{10} + \cdots \\
b_{10} &= -\frac{693}{1310720}e^{10} - \cdots
\end{aligned}
\tag{12}
$$

or

$$
M = \frac{a}{1+n}\left\{c_0\phi + c_2\sin 2\phi + c_4\sin 4\phi + c_6\sin 6\phi + c_8\sin 8\phi + c_{10}\sin 10\phi + \cdots\right\} \tag{13}
$$

where the coefficients $\left\{c_n\right\}$ are to order n^5 as follows

$$
\begin{aligned}
&c_0 = 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \cdots \quad c_2 = -\frac{3}{2}n + \frac{3}{16}n^3 + \frac{3}{128}n^5 + \cdots \quad c_4 = \frac{15}{16}n^2 - \frac{15}{64}n^4 + \cdots \\
&c_6 = -\frac{35}{48}n^3 + \frac{175}{768}n^5 + \cdots \quad c_8 = \frac{315}{512}n^4 + \cdots \qquad\qquad c_{10} = -\frac{693}{1280}n^5 + \cdots
\end{aligned}
\tag{14}
$$

Note here that for WGS84 ellipsoid (the reference ellipsoid of the World Geodetic System 1984) where $a = 6378137$ m and $f = 1/298.257223563$ the ellipsoid constants $n = 1.679220386383705e-003$ and $e^2 = 6.694379990141317e-003$, and $n^5 \cong \dfrac{e^{10}}{1007} \cong \dfrac{e^{12}}{6.7}$. This demonstrates that the series (13) with fewer terms in the coefficients $\left\{c_n\right\}$ is at least as 'accurate' as the series (11). To test this consider the meridian distance expressed as a sum of terms $M = M_0 + M_2 + M_4 + \cdots$ where for series (11) $M_0 = a\left(1-e^2\right)b_0\phi$, $M_2 = a\left(1-e^2\right)b_2\sin 2\phi$, $M_4 = a\left(1-e^2\right)b_4\sin 4\phi$, etc. and for series (13) $M_0 = \dfrac{a}{1+n}c_0\phi$, $M_2 = \dfrac{a}{1+n}c_2\sin 2\phi$, $M_4 = \dfrac{a}{1+n}c_4\sin 4\phi$, etc. Maximum values for $M_0, M_2, M_4, \cdots$ occur at latitudes $\phi = 90^\circ, 45^\circ, 22.5^\circ, \ldots$ when $\phi = \max$ or $\sin k\phi = 1$ and testing the differences between terms at these maximums revealed no differences greater than 0.5 micrometres. So series (13) should be the preferable method of computation. Indeed, further truncation of the coefficients $\left\{c_n\right\}$ to order n^4 and truncating series (13) at $c_8\sin 8\phi$ revealed no differences greater than 1 micrometre.

Quadrant length Q

The quadrant length of the ellipsoid Q is the length of the meridian arc from the equator to the pole and is obtained from equation (11) by setting $\phi = \tfrac{1}{2}\pi$, and noting that $\sin 2\phi$, $\sin 4\phi$, $\sin 6\phi$, … all equal zero, giving

$$\begin{aligned} Q &= a\left(1-e^2\right)b_0\,\tfrac{\pi}{2} \\ &= a\left(1-e^2\right)\left\{1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\frac{11025}{16384}e^8+\frac{43659}{65536}e^{10}+\cdots\right\}\frac{\pi}{2} \end{aligned} \tag{15}$$

Similarly, using equation (13)

$$Q = \frac{a}{1+n}c_0\,\tfrac{\pi}{2} = \frac{a}{1+n}\left\{1+\frac{1}{4}n^2+\frac{1}{64}n^4+\cdots\right\}\frac{\pi}{2} \tag{16}$$

Rectifying latitude μ and rectifying radius A

If the meridian distance M on the ellipsoid is equivalent to a meridian distance (great circle arc) on a (rectifying) sphere of radius A then the rectifying latitude μ is defined by

$$M = A\mu \tag{17}$$

An expression for A is obtained by considering the case when $\mu = \tfrac{1}{2}\pi$ and M is equal to the quadrant distance Q and (17) may be rearranged to give $A = 2Q/\pi$ and then using (15) to give A to order e^{10} as

$$\begin{aligned} A &= a\left(1-e^2\right)b_0 \\ &= a\left(1-e^2\right)\left\{1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\frac{11025}{16384}e^8+\frac{43659}{65536}e^{10}+\cdots\right\} \end{aligned} \tag{18}$$

Similarly, using (16)

$$A = \frac{a}{1+n}\left\{1+\frac{1}{4}n^2+\frac{1}{64}n^4+\cdots\right\} \tag{19}$$

Re-arranging (17) and using (11) and (18) gives the rectifying latitude as

$$\mu = M/A = \phi + \left(b_2/b_0\right)\sin 2\phi + \left(b_4/b_0\right)\sin 4\phi + \left(b_6/b_0\right)\sin 6\phi + \cdots \tag{20}$$

or

$$\mu = \phi + g_2\sin 2\phi + g_4\sin 4\phi + g_6\sin 6\phi + g_8\sin 8\phi + g_{10}\sin 10\phi + \cdots \tag{21}$$

where the coefficients $\left\{g_n = b_n/b_0\right\}$ are to order e^{10}

$$
\begin{aligned}
g_2 &= -\frac{3}{8}e^2 - \frac{3}{16}e^4 - \frac{111}{1024}e^6 - \frac{141}{2048}e^8 - \frac{1533}{32768}e^{10} + \cdots \\
g_4 &= \frac{15}{256}e^4 + \frac{15}{256}e^6 + \frac{405}{8192}e^8 + \frac{165}{4096}e^{10} + \cdots \\
g_6 &= -\frac{35}{3072}e^6 - \frac{35}{2048}e^8 - \frac{4935}{262144}e^{10} - \cdots \\
g_8 &= \frac{315}{131072}e^8 + \frac{315}{65536}e^{10} + \cdots \qquad g_{10} = -\frac{693}{1310720}e^{10} - \cdots
\end{aligned}
\tag{22}
$$

Similarly, we may obtain

$$
\mu = \phi + d_2 \sin 2\phi + d_4 \sin 4\phi + d_6 \sin 6\phi + d_8 \sin 8\phi + d_{10} \sin 10\phi + \cdots \tag{23}
$$

where the coefficients $\{d_n\}$ are to order n^5 as follows

$$
\begin{aligned}
&d_2 = -\frac{3}{2}n + \frac{9}{16}n^3 - \frac{3}{32}n^5 + \cdots \quad d_4 = \frac{15}{16}n^2 - \frac{15}{32}n^4 + \cdots \quad d_6 = -\frac{35}{48}n^3 + \frac{105}{256}n^5 - \cdots \\
&d_8 = \frac{315}{512}n^4 - \cdots \qquad d_{10} = -\frac{693}{1280}n^5 + \cdots
\end{aligned}
\tag{24}
$$

Maxima can easily perform the algebra required for (21) and (23) by using a Taylor series representation of b_0^{-1} for the 'e-series' and c_0^{-1} for the 'n-series'.

Latitude ϕ as a function of rectifying latitude μ

An expression for ϕ as a function of μ and powers of e is obtained by reversion of series (21) using Lagrange's theorem (see Appendix 3) giving

$$
\phi = \mu + G_2 \sin 2\mu + G_4 \sin 4\mu + G_6 \sin 6\mu + G_8 \sin 8\mu + G_{10} \sin 10\mu + \cdots \tag{25}
$$

where the coefficients $\{G_n\}$ are to order e^{10} as follows

$$
\begin{aligned}
G_2 &= \frac{3}{8}e^2 + \frac{3}{16}e^4 + \frac{213}{2048}e^6 + \frac{255}{4096}e^8 + \frac{20861}{524288}e^{10} + \cdots \\
G_4 &= \frac{21}{256}e^4 + \frac{21}{256}e^6 + \frac{533}{8192}e^8 + \frac{197}{4096}e^{10} + \cdots \\
G_6 &= \frac{151}{6144}e^6 + \frac{151}{4096}e^8 + \frac{5019}{131072}e^{10} + \cdots \\
G_8 &= \frac{1097}{65536}e^8 + \frac{1097}{131072}e^{10} + \cdots \qquad G_{10} = \frac{8011}{2621440}e^{10} + \cdots
\end{aligned}
\tag{26}
$$

Similarly, reversion of series (23) gives

$$
\phi = \mu + D_2 \sin 2\mu + D_4 \sin 4\mu + D_6 \sin 6\mu + D_8 \sin 8\mu + D_{10} \sin 10\mu + \cdots \tag{27}
$$

where the coefficients $\{D_n\}$ are to order n^5 as follows

$$D_2=\frac{3}{2}n-\frac{27}{32}n^3+\frac{269}{512}n^5-\cdots \quad D_4=\frac{21}{16}n^2-\frac{55}{32}n^4+\cdots \quad D_6=\frac{151}{96}n^3-\frac{417}{128}n^5+\cdots$$
$$D_8=\frac{1097}{512}n^4-\cdots \qquad D_{10}=\frac{8011}{2560}n^5-\cdots \tag{28}$$

Again, Maxima can easily perform the calculus and algebra required to obtain 'e-series' (25) or 'n-series' (27) using Lagrange's theorem.

The series developed above can be used in the direct and inverse problems on the ellipsoid using the great elliptic arc.

EQUATION OF A GREAT ELLIPTIC ARC SECTION

Figure 1 shows P on the great elliptic arc that passes through P_1 and P_2 on the ellipsoid. Parallels of latitude ϕ and meridians of longitude λ have their respective reference planes; the equator and the Greenwich meridian, and longitudes are measured 0° to $\pm 180^\circ$ (east positive, west negative) from the Greenwich meridian and latitudes are measured 0° to $\pm 90^\circ$ (north positive, south negative) from the equator. The x,y,z geocentric Cartesian coordinate system has an origin at O, the centre of the ellipsoid, and the z-axis is the minor axis (axis of revolution). The xOz plane is the Greenwich meridian plane (the origin of longitudes) and the xOy plane is the equatorial plane. The positive x-axis passes through the intersection of the Greenwich meridian and the equator, the positive y-axis is advanced 90° east along the equator and the positive z-axis passes through the north pole of the ellipsoid.

In Figure 1, θ_P is the geocentric latitude of P and (geodetic) latitude ϕ and geocentric latitude θ are related by

$$\tan\theta=\left(1-e^2\right)\tan\phi=\frac{b^2}{a^2}\tan\phi=\left(1-f\right)^2\tan\phi \tag{29}$$

The geometric relationship between geocentric latitude θ and (geodetic) latitude ϕ is shown in Figure 2, noting that the normal to the ellipsoid does not pass through the centre O but instead cuts the rotation axis of the ellipsoid at H and the distance $OH=\nu e^2\sin\phi$ and $\nu=PH$ is the radius of curvature in the prime vertical plane.

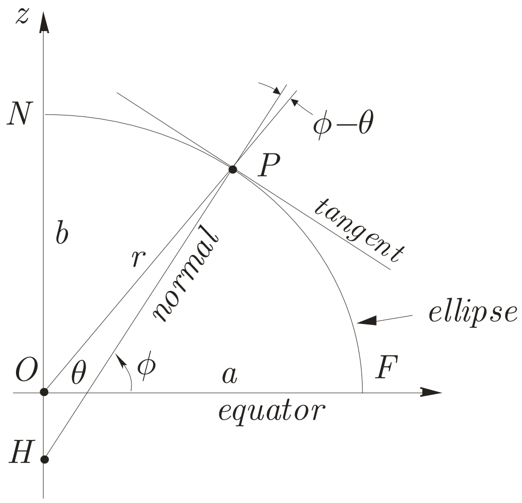

Figure 2: Meridian plane of P

The distance $r = OP$ will be useful and is given by the polar equation for an ellipse as

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \tag{30}$$

And after some algebra and the use of equations (3) and (4) we may obtain

$$r = \frac{b}{\sqrt{1 - e^2 \cos^2 \theta}} = \frac{a}{\sqrt{1 + e'^2 \sin^2 \theta}} \tag{31}$$

The great elliptic plane in Figure 1 is defined by points ①, ② and ③ that are P_1, P_2 and the centre of the ellipsoid O respectively. Cartesian coordinates of ① and ② are computed from the following equations

$$\begin{aligned} x &= \nu \cos\phi \cos\lambda &&= \frac{c}{V} \cos\phi \cos\lambda &&= r \cos\theta \cos\lambda \\ y &= \nu \cos\phi \sin\lambda &&= \frac{c}{V} \cos\phi \sin\lambda &&= r \cos\theta \sin\lambda \\ z &= \nu\left(1 - e^2\right) \sin\phi &&= \frac{b}{V} \sin\phi &&= r \sin\theta \end{aligned} \tag{32}$$

The Cartesian coordinates of point ③ are all zero.

The general equation of a plane may be written as

$$Ax + By + Cz + D = 0 \tag{33}$$

And the equation of the plane passing through points ①, ② and ③ is given in the form of a 3rd-order determinant

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \end{vmatrix} = 0 \tag{34}$$

or expanded into 2nd-order determinants

$$\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_2 & z_3-z_2 \end{vmatrix}\left(x-x_1\right)-\begin{vmatrix} x_2-x_1 & z_2-z_1 \\ x_3-x_2 & z_3-z_2 \end{vmatrix}\left(y-y_1\right)+\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_2 & y_3-y_2 \end{vmatrix}\left(z-z_1\right)=0 \quad (35)$$

Expanding the determinants in equation (35) gives

$$\begin{aligned} &\left(x-x_1\right)\left\{\left(y_2-y_1\right)\left(z_3-z_2\right)-\left(z_2-z_1\right)\left(y_3-y_2\right)\right\} \\ &-\left(y-y_1\right)\left\{\left(x_2-x_1\right)\left(z_3-z_2\right)-\left(z_2-z_1\right)\left(x_3-x_2\right)\right\} \\ &+\left(z-z_1\right)\left\{\left(x_2-x_1\right)\left(y_3-y_2\right)-\left(y_2-y_1\right)\left(x_3-x_2\right)\right\}=0 \end{aligned} \quad (36)$$

Now since $x_3 = y_3 = z_3 = 0$ equation (36) becomes

$$\begin{aligned} &\left(x-x_1\right)\left\{\left(y_2-y_1\right)\left(-z_2\right)-\left(z_2-z_1\right)\left(-y_2\right)\right\} \\ &-\left(y-y_1\right)\left\{\left(x_2-x_1\right)\left(-z_2\right)-\left(z_2-z_1\right)\left(-x_2\right)\right\} \\ &+\left(z-z_1\right)\left\{\left(x_2-x_1\right)\left(-y_2\right)-\left(y_2-y_1\right)\left(-x_2\right)\right\}=0 \end{aligned} \quad (37)$$

Expanding and simplifying equation (37) gives

$$x\left(y_1z_2-y_2z_1\right)-y\left(x_1z_2-x_2z_1\right)+z\left(x_1y_2-x_2y_1\right)=0 \quad (38)$$

or

$$Ax-By+Cz=0 \quad (39)$$

where A, B and C are functions of the coordinates of the terminal points P_1 and P_2

$$A=y_1z_2-y_2z_1 \qquad B=x_1z_2-x_2z_1 \qquad C=x_1y_2-x_2y_1 \quad (40)$$

Replacing x, y and z with their equivalents, given by equations (32), gives

$$\nu\cos\phi\cos\lambda\left(y_1z_2-y_2z_1\right)-\nu\cos\phi\sin\lambda\left(x_1z_2-x_2z_1\right)+\nu\left(1-e^2\right)\sin\phi\left(x_1y_2-x_2y_1\right)=0$$

and dividing both sides by $\nu\cos\phi$ gives the equation of the great elliptic as

$$A\cos\lambda-B\sin\lambda+C\left(1-e^2\right)\tan\phi=0 \quad (41)$$

Equation (41) is not suitable for computing the distance along a great elliptic arc, nor is it suitable for computing the azimuth of the curve, but by certain re-arrangements it is possible to solve (directly) for the latitude of a point on the curve given a longitude somewhere between the longitudes of the terminal points of the curve. Or alternatively, solve (iteratively) for the longitude of a point given a latitude somewhere between the latitudes of the terminal points.

Solving for the latitude

A simple re-arrangement of equation (41) allows the latitude ϕ to be evaluated from

$$\tan\phi = \frac{B\sin\lambda - A\cos\lambda}{C\left(1-e^2\right)} \tag{42}$$

where A and B and C are functions of terminal points P_1 and P_2 given by equations (40).

Solving for the longitude

The longitude λ can be evaluated using Newton-Raphson iteration for the real roots of the equation $f\left(\lambda\right) = 0$ given in the form of an iterative equation

$$\lambda_{(n+1)} = \lambda_{(n)} - \frac{f\left(\lambda_{(n)}\right)}{f'\left(\lambda_{(n)}\right)} \tag{43}$$

where n denotes the n^{th} iteration and $f\left(\lambda\right)$ is given by equation (41) as

$$f\left(\lambda\right) = A\cos\lambda - B\sin\lambda + C\left(1-e^2\right)\tan\phi \tag{44}$$

and the derivative $f'\left(\lambda\right) = \dfrac{d}{d\lambda}\left\{f\left(\lambda\right)\right\}$ is given by

$$f'\left(\lambda\right) = -A\sin\lambda - B\cos\lambda \tag{45}$$

An initial value of $\lambda_{(1)}$ (λ for $n=1$) can be taken as the longitude of P_1 and the functions $f\left(\lambda_{(1)}\right)$ and $f'\left(\lambda_{(1)}\right)$ evaluated from equations (44) and (45) using λ_1. $\lambda_{(2)}$ (λ for $n=2$) can now be computed from equation (43) and this process repeated to obtain values $\lambda_{(3)}, \lambda_{(4)}, \ldots$. This iterative process can be concluded when the difference between $\lambda_{(n+1)}$ and $\lambda_{(n)}$ reaches an acceptably small value.

Alternatively, the longitude can be evaluated by a trigonometric equation derived as follows. Equation (41) can be expressed as

$$B\sin\lambda - A\cos\lambda = C\left(1-e^2\right)\tan\phi \tag{46}$$

and A, B and C are given by equations (40). Equation (46) can be expressed as a trigonometric addition of the form

$$\begin{aligned} C\left(1-e^2\right)\tan\phi &= q\cos\left(\lambda-\delta\right) \\ &= q\cos\lambda\cos\delta + q\sin\lambda\sin\delta \end{aligned} \tag{47}$$

Now, equating the coefficients of $\cos\lambda$ and $\sin\lambda$ in equations (47) and (46) gives

$$A = -q\cos\delta; \quad B = q\sin\delta \tag{48}$$

and using these relationships

$$q = \sqrt{A^2 + B^2}; \quad \tan\delta = \frac{B}{-A} \tag{49}$$

Substituting these results into equation (47) gives

$$\lambda = \arccos\left\{\frac{C\left(1-e^2\right)\tan\phi}{\sqrt{A^2 + B^2}}\right\} + \arctan\left\{\frac{B}{-A}\right\} \tag{50}$$

THE VERTEX OF THE GREAT ELLIPTIC

As P moves along a great elliptic its latitude and longitude is varying (unless the great elliptic is the equator or a meridian, in which case either latitude or longitude will be constant) and differentiating equation (41) with respect to latitude ϕ and re-arranging gives the derivative

$$\frac{d\phi}{d\lambda} = \frac{A\sin\lambda + B\cos\lambda}{C\left(1-e^2\right)\sec^2\phi} \tag{51}$$

Now when $\dfrac{d\phi}{d\lambda} = 0$ the latitude will be a maximum and P will have reached a vertex ϕ_V, λ_V and (51) becomes $\dfrac{d\phi}{d\lambda} = 0 = A\sin\lambda_V + B\cos\lambda_V$ from which we obtain

$$\lambda_V = \arctan\left\{\frac{B}{-A}\right\} \tag{52}$$

And substituting (52) into (41) and re-arranging gives

$$\phi_V = \arctan\left\{\frac{B\sin\lambda_V - A\cos\lambda_V}{C\left(1-e^2\right)}\right\} \tag{53}$$

Using (29) gives the geocentric latitude of the vertex as

$$\theta_V = \arctan\left\{\frac{B\sin\lambda_V - A\cos\lambda_V}{C}\right\} \tag{54}$$

GEOMETRIC PARAMETERS OF THE GREAT ELLIPTIC

The semi-major axis of the great elliptic is a, the radius of the equator and the semi-major axis of the ellipsoid. The semi-minor axis of the great elliptic is the distance $OV = b'$ (see Figure 1) and an expression for b' is obtained from (31) [the polar equation for an ellipse]

$$b' = \frac{a}{\sqrt{1 + e'^2 \sin^2 \theta_V}} = \frac{b}{\sqrt{1 - e^2 \cos^2 \theta_V}} \tag{55}$$

Denoting ε and ε' as the first and second eccentricities of the great elliptic (having semi-axes a, b' and $a > b'$) ε and ε' are defined by

$$\varepsilon^2 = \frac{a^2 - b'^2}{a^2} \tag{56}$$

$$\varepsilon'^2 = \frac{a^2 - b'^2}{b'^2} \tag{57}$$

And using (55) and (56) the second eccentricity of the great elliptic is

$$\varepsilon'^2 = e'^2 \sin^2 \theta_V \tag{58}$$

THE AZIMUTH α OF THE GREAT ELLIPTIC

The azimuth α of a great elliptic at P (α_1 at P_1 in Figure 1) is the clockwise angle between the tangent to the ellipsoid meridian and the tangent to the great elliptic. Both tangents lie in the tangent plane to the ellipsoid at P. The forward azimuth of the great elliptic from P_1 to P_2 is denoted α_{12} and $\alpha_{12} = \alpha_1$ (see Figure 1) The reverse azimuth (P_2 to P_1) is denoted α_{21} and $\alpha_{21} = \alpha_2 + 180^\circ$.

The azimuth can be obtained by a series of vector manipulations (scalar and vector products; see Appendix 1) that are outlined below.

The position vector of P on the surface of the ellipsoid is (Deakin & Hunter 2010a)

$$\mathbf{r} = \mathbf{r}\left(\phi, \lambda\right) = \frac{c}{V} \cos\phi \cos\lambda \,\mathbf{i} + \frac{c}{V} \cos\phi \sin\lambda \,\mathbf{j} + \frac{b}{V} \sin\phi \,\mathbf{k} \tag{59}$$

and the partial derivatives of the position vector are

$$\mathbf{r}_\phi = \frac{\partial}{\partial\phi} \mathbf{r}\left(\phi, \lambda\right) = -\frac{c}{V^3} \sin\phi \cos\lambda \,\mathbf{i} - \frac{c}{V^3} \sin\phi \sin\lambda \,\mathbf{j} + \frac{c}{V^3} \cos\phi \,\mathbf{k} \tag{60}$$

$$\mathbf{r}_\lambda = \frac{\partial}{\partial\lambda} \mathbf{r}\left(\phi, \lambda\right) = -\frac{c}{V} \cos\phi \sin\lambda \,\mathbf{i} + \frac{c}{V} \cos\phi \cos\lambda \,\mathbf{j} + 0 \,\mathbf{k} \tag{61}$$

where $\mathbf{r}_\phi$ and $\mathbf{r}_\lambda$ are orthogonal and tangential to the parametric curves $\lambda = \text{constant}$ (meridian) and $\phi = \text{constant}$ (parallel). $\mathbf{r}_\phi$ points north and $\mathbf{r}_\lambda$ points east and both vectors lie in the tangent plane to the ellipsoid at P.

Now, denote two unit vectors pointing north and east respectively as $\hat{\mathbf{n}} = \dfrac{\mathbf{r}_\phi}{\left|\mathbf{r}_\phi\right|}$ and $\hat{\mathbf{e}} = \dfrac{\mathbf{r}_\lambda}{\left|\mathbf{r}_\lambda\right|}$ and

$$\hat{\mathbf{n}} = -\sin\phi\cos\lambda\,\mathbf{i} - \sin\phi\sin\lambda\,\mathbf{j} + \cos\phi\,\mathbf{k} \tag{62}$$

$$\hat{\mathbf{e}} = -\sin\lambda\,\mathbf{i} + \cos\lambda\,\mathbf{j} + 0\,\mathbf{k} \tag{63}$$

The unit vector normal to the surface (pointing inwards) is denoted $\hat{\mathbf{N}}$ and found from the vector cross product as

$$\hat{\mathbf{N}} = \hat{\mathbf{n}} \times \hat{\mathbf{e}} = -\cos\phi\cos\lambda\,\mathbf{i} - \cos\phi\sin\lambda\,\mathbf{j} - \sin\phi\,\mathbf{k} \tag{64}$$

$\hat{\mathbf{n}}$, $\hat{\mathbf{e}}$ and $\hat{\mathbf{N}}$ form a triplet of orthogonal unit vectors.

Now, the great elliptic plane is given by (39) as $Ax - By + Cz = 0$ and the unit vector normal to this plane (directed outwards from the origin) is denoted by $\hat{\mathbf{p}}$ and

$$\hat{\mathbf{p}} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k} \tag{65}$$

where l, m and n are direction cosines ($l^2 + m^2 + n^2 = 1$) given by

$$l = \frac{A}{\sqrt{A^2 + B^2 + C^2}}; \quad m = \frac{-B}{\sqrt{A^2 + B^2 + C^2}}; \quad n = \frac{C}{\sqrt{A^2 + B^2 + C^2}} \tag{66}$$

and A, B and C are constants of the great elliptic given by (40).

Now the vector cross product of the two unit vectors $\hat{\mathbf{N}}$ and $\hat{\mathbf{p}}$ at P_1 will produce a vector perpendicular to both and in the direction to P_2. This vector denoted $\mathbf{g}$ (and its unit vector $\hat{\mathbf{g}}$) will lie in the tangent plane at P_1 and

$$\mathbf{g} = \hat{\mathbf{N}} \times \hat{\mathbf{p}} \quad \text{and} \quad \hat{\mathbf{g}} = \frac{\mathbf{g}}{\left|\mathbf{g}\right|} \tag{67}$$

The vector dot product can now be used to compute angles α and β in the tangent plane; where α is the angle between the unit vectors $\hat{\mathbf{n}}$ (pointing north) and $\hat{\mathbf{g}}$

$$\alpha = \arccos\left\{\hat{\mathbf{n}} \bullet \hat{\mathbf{g}}\right\} \tag{68}$$

and β is the angle between the unit vector $\hat{\mathbf{e}}$ (pointing east) and $\hat{\mathbf{g}}$

$$\beta = \arccos\left\{\hat{\mathbf{e}} \bullet \hat{\mathbf{g}}\right\} \tag{69}$$

The azimuth $\alpha_1 = \alpha_{12}$ of the great elliptic from P_1 to P_2 is then determined from the rule:

$$\text{IF } \beta > 90^\circ \text{ THEN } \alpha_{12} = 360^\circ - \alpha \text{ ELSE } \alpha_{12} = \alpha \tag{70}$$

THE AZIMUTH α OF THE GREAT ELLIPTIC – THOMAS' APPROACH

The American mathematician Paul D. Thomas (Thomas 1965) developed a trigonometric formula for azimuth, based on vector manipulations, which is outlined here. Consider a rotated Cartesian frame x', y', z' where the x'-y' plane is the plane of the equator, the z'-axis is coincident with z-axis and the z'-x' plane is the meridian plane of P_1. The Cartesian coordinates of P_1 and P_2 in this rotated system are

$$\begin{aligned} x_1' &= \frac{c}{V_1}\cos\phi_1 \\ y_1' &= 0 \\ z_1' &= \frac{b}{V_1}\sin\phi_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_2' &= \frac{c}{V_2}\cos\phi_2 \cos\Delta\lambda \\ y_2' &= \frac{c}{V_2}\cos\phi_2 \sin\Delta\lambda \\ z_2' &= \frac{b}{V_2}\sin\phi_2 \end{aligned} \tag{71}$$

where $\Delta\lambda = \lambda_2 - \lambda_1$, c is the polar radius and V is a latitude function (see equations (5), (6) and (7)). Substituting these expressions into equation (38) and simplifying gives the great elliptic plane as

$$A'x' - B'y' + C'z' = 0 \tag{72}$$

where

$$A' = -\left(1 - e^2\right)\tan\phi_1 \sin\Delta\lambda \quad B' = \left(1 - e^2\right)\left(\tan\phi_2 - \tan\phi_1 \cos\Delta\lambda\right) \quad C' = \sin\Delta\lambda \tag{73}$$

The unit vector perpendicular to the great elliptic plane (pointing outwards from the origin) is $\hat{\mathbf{p}}' = \frac{A'}{d}\mathbf{i} + \frac{-B'}{d}\mathbf{j} + \frac{C'}{d}\mathbf{k}$ where $d = \sqrt{A'^2 + B'^2 + C'^2}$ and the unit vector normal to the tangent plane at P_1 (pointing inwards) is $\hat{\mathbf{N}}_1' = -\cos\phi_1\,\mathbf{i} - 0\,\mathbf{j} - \sin\phi_1\,\mathbf{k}$. The tangent to the meridian at P_1 (pointing north) is $\hat{\mathbf{n}}_1' = -\sin\phi_1\,\mathbf{i} - 0\,\mathbf{j} + \cos\phi_1\,\mathbf{k}$ and similarly to before, the vector tangential to the great elliptic and in the tangent plane at P_1 is $\mathbf{g}_1' = \hat{\mathbf{N}}_1' \times \hat{\mathbf{p}}' = -\frac{B'}{d}\sin\phi_1\,\mathbf{i} + \left(\frac{C'}{d}\cos\phi_1 + \frac{A'}{d}\sin\phi_1\right)\mathbf{j} + \frac{B'}{d}\cos\phi_1\,\mathbf{k}$ having magnitude $\left|\mathbf{g}'\right| = \frac{1}{d}\sqrt{B'^2 + \left(C'\cos\phi_1 - A'\sin\phi_1\right)^2}$ and unit vector $\hat{\mathbf{g}}' = \frac{\mathbf{g}'}{\left|\mathbf{g}'\right|}$.

Using unit vectors, $\cos\alpha_1 = \hat{\mathbf{n}}_1' \bullet \hat{\mathbf{g}}'$ which after some algebra reduces to

$\cos\alpha_1 = \dfrac{B'}{\sqrt{B'^2 + \left(C'\cos\phi_1 - A'\sin\phi_1\right)^2}}$ and with the trigonometric identity

$\sin^2 x + \cos^2 x = 1$ we obtain $\sin\alpha_1 = \dfrac{C'\cos\phi_1 - A'\sin\phi_1}{\sqrt{B'^2 + \left(C'\cos\phi_1 - A'\sin\phi_1\right)^2}}$; and then

$\tan\alpha_1 = \dfrac{C'\cos\phi_1 - A'\sin\phi_1}{B'}$. Substituting expressions for A', B', C' given by (73) and simplifying gives

$$\alpha_1 = \arctan\left\{\frac{a^2\sin\Delta\lambda}{\nu_1^2\left(1-e^2\right)\left(\tan\phi_2 - \tan\phi_1\cos\Delta\lambda\right)\cos\phi_1}\right\} \tag{74}$$

Evaluating α_1 in the range $-180^\circ \le \alpha_1 \le 180^\circ$ gives the azimuth α_{12} from the rule:

$$\text{IF } \alpha_1 < 0 \text{ THEN } \alpha_{12} = 360^\circ + \alpha_1 \text{ ELSE } \alpha_{12} = \alpha_1 \tag{75}$$

Following Thomas (1965); by symmetrical interchange of subscripts and replacing $\Delta\lambda$ by $-\Delta\lambda$ the azimuth from P_2 to P_1 is

$$\alpha_2 = \arctan\left\{\frac{a^2\sin\Delta\lambda}{\nu_2^2\left(1-e^2\right)\left(\tan\phi_2\cos\Delta\lambda - \tan\phi_1\right)\cos\phi_2}\right\} \tag{76}$$

Evaluating α_2 in the range $-180^\circ \le \alpha_2 \le 180^\circ$ gives the azimuth α_{21} from the rule:

$$\text{IF } \alpha_2 < 0 \text{ THEN } \alpha_{21} = 540^\circ + \alpha_2 \text{ ELSE } \alpha_{21} = \alpha_2 + 180^\circ \tag{77}$$

[Note that Thomas' equations for azimuth (Thomas 1965, p. 47, equations (15),(16)) are different from (74) and (76) as Thomas defines azimuth as a clockwise angle measured from south, see p. 45, equation (9)]

GREAT ELLIPTIC LATITUDES

A point P on the great elliptic has 'great elliptic' latitudes ϕ' and θ' that are analogous to (geodetic) latitude ϕ and geocentric latitude θ noting that the great elliptic arc *EPV* of Figure 1 is similar to the meridian arc *FPN.* In fact we could consider the vertex V of the great elliptic as analogous to the pole N of a meridian. Expressions for ϕ' and θ' are developed below.

In Figure 1, consider two vectors $\mathbf{E} = \overrightarrow{OE} = x_E\,\mathbf{i} + y_E\,\mathbf{j} + z_E\,\mathbf{j}$ and $\mathbf{P} = \overrightarrow{OP} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{j}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the direction of the Cartesian axes and x_E, y_E, z_E and x, y, z are Cartesian coordinates of E and P respectively. The vector dot product is

$$\mathbf{E} \bullet \mathbf{P} = \left|\mathbf{E}\right|\left|\mathbf{P}\right|\cos\theta' = x_E x + y_E y + z_E z \tag{78}$$

and a re-arrangement of (78) gives an expression for great elliptic geocentric latitude θ' as

$$\cos\theta' = \frac{x_E x + y_E y + z_E z}{\left|\mathbf{E}\right|\left|\mathbf{P}\right|} \tag{79}$$

where the Cartesian coordinates are obtained from (32) as

$$\begin{aligned} x_E &= a\cos\lambda_E & & x = \nu\cos\phi\cos\lambda \\ y_E &= a\sin\lambda_E \quad & \text{and} \quad & y = \nu\cos\phi\sin\lambda \\ z_E &= 0 & & z = \nu\left(1 - e^2\right)\sin\phi \end{aligned} \tag{80}$$

$\left|\mathbf{E}\right| = \sqrt{x_E^2 + y_E^2 + z_E^2}$ and $\left|\mathbf{P}_1\right| = \sqrt{x^2 + y^2 + z^2}$ are the magnitudes of vectors **E** and **P**.

Now $\left|\mathbf{E}\right|$ is just the semi-major axis a; and $\left|\mathbf{P}\right| = \frac{a}{W}\sqrt{W^2\left(2 - e^2\right) - \left(1 - e^2\right)}$ is obtained after some algebra noting that W is a latitude function defined by (7). Substituting these results and (80) into (79) and simplifying gives

$$\theta' = \arccos\left\{\frac{\cos\phi\cos\left(\lambda - \lambda_E\right)}{\sqrt{W^2\left(2 - e^2\right) - \left(1 - e^2\right)}}\right\} \tag{81}$$

And using the relationships between latitudes given in (29) we have

$$\phi' = \arctan\left\{\frac{\tan\theta'}{1 - \varepsilon^2}\right\} \tag{82}$$

THE AUXILIARY SPHERE AND THE GREAT ELLIPTIC

Following Bowring (1984) several formula, useful in the solution of the direct and inverse problems, can be developed using spherical trigonometry and an auxiliary sphere. This auxiliary sphere has a radius $r_1 = OP_1\left(b \le r_1 \le a\right)$ and is centred at O the centre of the ellipsoid.

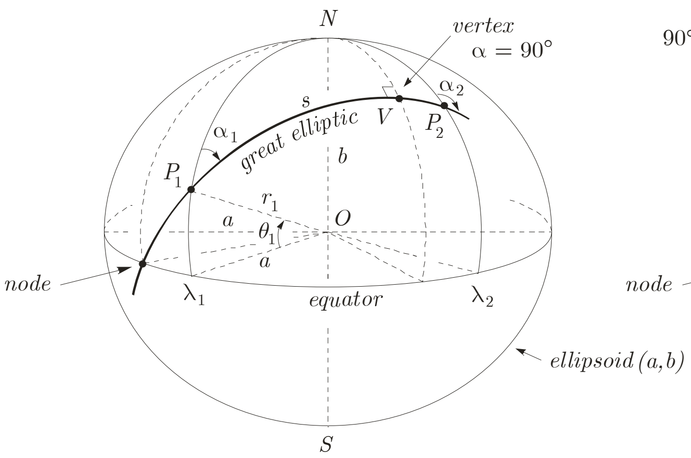

Figure 3a:

Great elliptic through P_1 and P_2

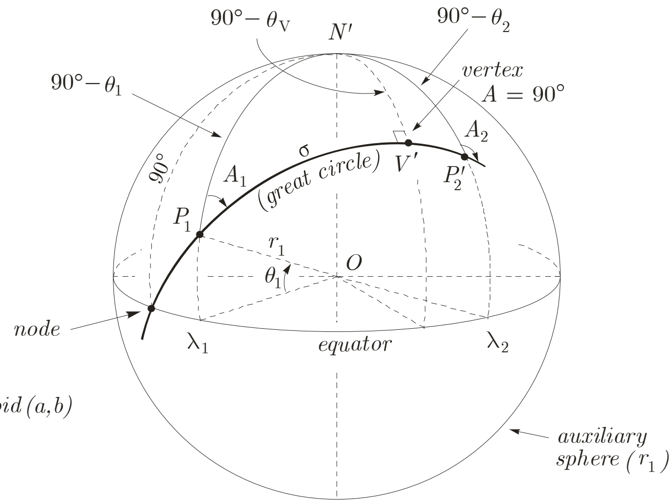

Figure 3b:

Great circle through P_1 and P_2'

Figure 3a shows the great elliptic arc $s = P_1P_2$ on the ellipsoid having (forward) azimuth $\alpha_{12} = \alpha_1$ and reverse azimuth $\alpha_{21} = \alpha_2 + 180^\circ$; and azimuth $\alpha = 90^\circ$ at the vertex V. Figure 3b shows the great elliptic plane OP_1P_2 intersecting the auxiliary sphere along the great circle arc $P_1V'P_2'$. The great circle arc of length $\sigma = P_1P_2'$ has spherical azimuth A_1 at P_1, A_2 at P_2' and $A = 90^\circ$ at the vertex V'. And points P_1, V' and P_2' have spherical latitudes equal to the geocentric latitudes of P_1, V and P_2 on the ellipsoid.

In the spherical triangle $P_1N'P_2'$ in Figure 3b with $\Delta\lambda = \lambda_2 - \lambda_1$ the cotangent formula (four-parts rule) of spherical trigonometry may be used to give expressions for spherical azimuth A_1 (Bowring 1984) and longitude difference $\Delta\lambda$

$$A_1 = \arctan\left\{\frac{\sin\Delta\lambda}{\cos\theta_1\tan\theta_2 - \sin\theta_1\cos\Delta\lambda}\right\} \tag{83}$$

$$\Delta\lambda = \arctan\left\{\frac{\sin A_1}{\dfrac{\cos\theta_1}{\tan\sigma} - \sin\theta_1\cos A_1}\right\} \tag{84}$$

In the same spherical triangle, the cosine formula for spherical trigonometry gives expressions for geocentric latitude θ_2 and spherical arc length σ

$$\theta_2 = \arcsin\left(\sin\theta_1\cos\sigma + \cos\theta_1\sin\sigma\cos A_1\right) \tag{85}$$

$$\sigma = \arccos\left(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\cos\Delta\lambda\right) \tag{86}$$

When distances are small the spherical angles σ and $\Delta\lambda$ will be small and equation (86) may give unreliable results. Two better formula for evaluating σ are:

$$\sigma = \arcsin\sqrt{\left(\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \cos\Delta\lambda\right)^2 + \left(\cos\theta_2 \sin\Delta\lambda\right)^2} \tag{87}$$

and

$$\sigma = 2\arctan\sqrt{\frac{x}{1-x}} \tag{88}$$

where $x = \sin^2 \tfrac{1}{2}\Delta\theta + \cos\theta_1 \cos\theta_2 \sin^2 \tfrac{1}{2}\Delta\lambda$, $\Delta\theta = \theta_2 - \theta_1$ and $\Delta\lambda = \lambda_2 - \lambda_1$.

Equation (87) can be obtained from the five-parts cosine formula for spherical trigonometry and is used by Vincenty (1975, eq. 14, p. 90) in the equations for solving the inverse problem of the geodesic on the ellipsoid. Equation (88) is the 'haversine' formula (Sinnott 1984) where versine (versed sine) and haversine (half versine) are trigonometric functions defined as $\text{versine}\,\beta = 1 - \cos\beta$ and $\text{haversine}\,\beta = \tfrac{1}{2}\left(1 - \cos\beta\right) = \sin^2 \tfrac{1}{2}\beta$.

In the right-angled spherical triangle $P_1N'V'$ in Figure 3b we may use Napier's rules of circular parts to give expressions for geocentric latitude and the longitude of the vertex

$$\theta_V = \arccos\left(\sin A_1 \cos\theta_1\right) \tag{89}$$

$$\lambda_V = \arctan\left\{\frac{1}{\sin\theta_1 \tan A_1}\right\} + \lambda_1 \tag{90}$$

Bowring (1984) gives an expression for $\sin^2\theta_V$ that can be obtained from (89) using the trigonometric identity $\sin^2 x + \cos^2 x = 1$ to give

$$\sin^2\theta_V = \sin^2\theta_1 + \cos^2\theta_1 \cos^2 A_1 \tag{91}$$

Equation (91) can be used in (55) to compute the semi-major axis of the great elliptic.

RELATIONSHIP BETWEEN AZIMUTH α_1 AND SPHERICAL AZIMUTH A_1

The azimuth α of a great elliptic at P (α_1 at P_1 and α_2 at P_2 in Figure 3a) is the clockwise angle between the tangent to the ellipsoid meridian and the tangent to the great elliptic. Both these tangents lie in the tangent plane to the ellipsoid at P. The spherical azimuth A (A_1 at P_1 and A_2 at P_2' in Figure 3b) is an angle in a spherical triangle and is equal to the angle between the tangents to the intersecting great circle planes. These tangents lie in the tangent plane to the auxiliary sphere. At P_1, where the radius of the auxiliary sphere is $r_1 = OP_1$ the two tangent planes will intersect (the line of intersection is perpendicular to the meridian plane at P_1) and a relationship can be established between α_1 and A_1 by considering Figure 4

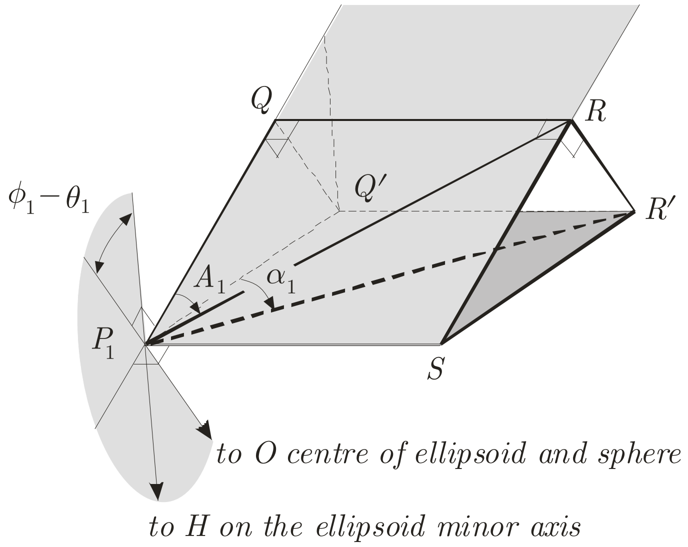

Figure 4: Tangent planes at P_1 on the ellipsoid

Figure 4 shows two tangent planes at P_1: (i) the plane P_1QRS that is tangential to the auxiliary sphere, and (ii) the plane $P_1Q'R'S$ tangential to the ellipsoid. The plane P_1RR' is coincident with the great elliptic plane and the great circle plane that both contain the centre of the ellipsoid O and the point P_2. P_1R and P_1R' are tangential to the great circle and the great elliptic respectively. The plane P_1QQ' is coincident with the meridian planes of the auxiliary sphere and the ellipsoid. And both planes contain the centre of the ellipsoid O, the point H on the minor axis and both poles N and N' (see Figures 2 and 3). The directions from P_1 to H and P_1 to O are the directions of the normal to the ellipsoid and the normal to the auxiliary sphere respectively and their difference $\phi_1 - \theta_1$ is the angle QP_1Q'. The spherical azimuth A_1 is the angle QP_1R and the azimuth α_1 is the angle $Q'P_1R'$.

In triangle P_1QR we have $\tan A_1 = \dfrac{QR}{P_1Q}$ and in triangle $P_1Q'R'$ we have $\tan\alpha_1 = \dfrac{Q'R'}{P_1Q'}$.

But, $QR = Q'R'$ and $P_1Q = P_1Q'\cos\left(\phi_1 - \theta_1\right)$, hence $\tan A_1 = \dfrac{Q'R'}{P_1Q'\cos\left(\phi_1 - \theta_1\right)}$ and

$$\tan\alpha_1 = \tan A_1\cos\left(\phi_1 - \theta_1\right) \tag{92}$$

Hence

$$\alpha_1 = \arctan\left\{\tan A_1\cos\left(\phi_1 - \theta_1\right)\right\} \tag{93}$$

$$A_1 = \arctan\left\{\frac{\tan\alpha_1}{\cos\left(\phi_1 - \theta_1\right)}\right\} \tag{94}$$

THE DIRECT PROBLEM ON THE ELLIPSOID USING A GREAT ELLIPTIC

The direct problem is: Given latitude and longitude of P_1, azimuth α_{12} of the great elliptic section P_1P_2 and the arc length s along the great elliptic curve; compute the latitude and longitude of P_2.

With the ellipsoid constants a, f, e^2 and e'^2 and given $\phi_1, \lambda_1, \alpha_{12}$ and s the problem may be solved by the following sequence.

1. Compute the geocentric latitude θ_1 at P_1 using equation (29) expressed as

$$\theta_1 = \arctan\left\{\left(1 - e^2\right)\tan\phi_1\right\}$$

2. Compute the spherical azimuth A_1 at P_1 using equation (94).

3. Compute the geocentric latitude θ_V and longitude λ_V of the vertex V and the latitude ϕ_V of the vertex and the longitude λ_E of the node E of the great elliptic (see Figures 1 and 3) using equations (89), (29) and (90)

4. Compute the geometric parameters of the great elliptic using equations (55) and equations (1) to (4) given here again as

$$b' = \frac{a}{\sqrt{1 + e'^2 \sin^2\theta_V}}; \quad f' = \frac{a - b'}{a}; \quad n' = \frac{f'}{2 - f'}; \quad \varepsilon^2 = f'\left(2 - f'\right); \quad \varepsilon'^2 = \frac{\varepsilon^2}{1 - \varepsilon^2}$$

5. Compute x,y,z Cartesian coordinates of P_1 using equations (6), (7) and (32) combined as

$$x_1 = \frac{c}{V_1}\cos\phi_1\cos\lambda_1; \quad y_1 = \frac{c}{V_1}\cos\phi_1\sin\lambda_1; \quad z_1 = \frac{b}{V_1}\sin\phi_1$$

6. Compute elliptic geocentric latitude θ_1' and elliptic latitude ϕ_1' of P_1 using equations (81) and (82)

7. Compute the meridian distance $M_1 = EP_1$ and the quadrant distance $Q = EV$ of the great elliptic using the series (13) with ϕ' and n' replacing ϕ and n. Then compute meridian distance $M_2 = 2Q - \left(M_1 \pm s\right)$ or $M_2 = M_1 \pm s$ depending on whether P_1

and P_2 are on opposite sides of the vertex (see Figure 1) or on the same side of the vertex.

8. Compute elliptic latitude ϕ_2' of P_2 using series (27) with ϕ' and n' replacing ϕ and n and then the elliptic geocentric latitude θ_2' from equation (29) expressed as

$$\theta_2' = \arctan\left\{\left(1-\varepsilon^2\right)\tan\phi_1'\right\}$$

9. Compute the great circle distance $\sigma = P_1P_2 = \left(90^\circ - \theta_1'\right) + \left(90^\circ - \theta_2'\right)$ on the auxiliary sphere.

10. Compute geocentric latitude θ_2 of P_2 from spherical trigonometry using equation (85) and then latitude ϕ_2 from equation (29) expressed as

$$\phi_2 = \arctan\left\{\frac{\tan\theta_2}{1-e^2}\right\}$$

11. Compute longitude difference $\Delta\lambda = \lambda_2 - \lambda_1$ from spherical trigonometry using equation (84) and then longitude of P_2

12. Compute reverse azimuth α_{21} using equation (76).

Shown below is the output of a MATLAB function *GEA_ direct.m* that solves the direct problem on the ellipsoid for great elliptic sections.

The ellipsoid is the WGS84 ellipsoid and ϕ,λ for P_1 are $+35^\circ\,45'\,55''$ and $+140^\circ\,23'\,08''$ respectively with $\alpha_{12} = 54^\circ\,57'\,06.932985''$ and $s = 8\,246\,278.910557\,\text{m}$. ϕ,λ computed for P_2 are $+37^\circ\,37'\,08''$ and $-122^\circ\,22'\,30''$ respectively.

```
>> GEA_direct

//////////////////////////////////
// Great Elliptic: Direct Case //
//////////////////////////////////

ellipsoid parameters
a   =  6378137.000000000
f   = 1/298.257223563000
b   =  6356752.314245179
c   =  6399593.625758493
e2  =  6.694379990141e-003
ep2 =  6.739496742276e-003
n   =  9.362215099742e-004

Latitude  P1 =   35 45 55.000000 (D M S)
Longitude P1 =  140 23  8.000000 (D M S)

Azimuth of Great Elliptic section P1-P2
Az12  =  54 57  6.932985  (D M S)
```

```
Great Elliptic section distance P1-P2
s =   8246278.910557

Cartesian coordinates
            X               Y               Z
P1   -3991399.691755  3303668.240372  3707090.313132

Latitude   V =   48 26 49.347671 (D M S)
Longitude  V = -169 17 28.736206 (D M S)
Longitude  E =  100 42 31.263794 (D M S)

t = lat-theta =    0 10 56.345218 (D M S)

spherical azimuth of Great Elliptic section P1-P2
A1    =  54 57  7.423927  (D M S)

Great Elliptic parameters
a   =  6378137.000000000
f   = 1/534.561645319167
b   =  6366205.472446818
e2  =  3.737883789191e-003
ep2 =  3.751907985228e-003
n   =  9.362215099742e-004

Great Elliptic Arc distances
M1 =  5702548.255834075
Q  = 10009385.364900846
M2 = 2*Q-(s+M1) =  6069943.563410141

Latitude  P2 =   37 37  8.000000 (D M S)
Longitude P2 = -122 22 30.000000 (D M S)

Azimuth of Great Elliptic section P2-P1 using Thomas' eq. 15
Az21  = 303  1 14.140673  (D M S)

>>
```

THE INVERSE PROBLEM ON THE ELLIPSOID USING A GREAT ELLIPTIC SECTION

The inverse problem is: Given latitudes and longitudes of P_1 and P_2 on the ellipsoid compute the azimuth α_{12} of the great elliptic section P_1P_2 and the arc length s of the great elliptic curve.

With the ellipsoid constants a, f, e^2 and e'^2 and given ϕ_1, λ_1 and ϕ_2, λ_2 the problem may be solved by the following sequence.

1. Compute Cartesian coordinates of P_1 and P_2 using equations (32).
2. Compute forward and reverse azimuths using equations (74) and (76)
3. Compute great elliptic constants A, B and C using equations (40)
4. Compute latitude and longitude of vertex using equations (52) and (53); and then the geocentric latitude of the vertex using (54)
5. Compute the geometric parameters of the great elliptic (see step 4 in the direct case)

6. Compute elliptic geocentric latitudes θ_1', θ_2' and elliptic latitudes ϕ_1', ϕ_2' of P_1 and P_2 using equations (81) and (82)

7. Compute great elliptic meridian distances $M_1 = EP_1$ and $M_2 = EP_2$ (or $M_2 = E'P_2$ depending on whether P_2 is on the other side of the vertex as in Figure 1) and the quadrant distance Q using the series (13).

8. Compute the great elliptic arc length from $s = \left|M_2 - M_1\right|$ or $s = 2Q - \left(M_1 + M_2\right)$

Shown below is the output of a MATLAB function *GEA_ inverse.m* that solves the inverse problem on the ellipsoid for great elliptic sections.

The ellipsoid is the WGS84 ellipsoid and ϕ, λ for P_1 are $+35^\circ\, 45'\, 55''$ and $+140^\circ\, 23'\, 08''$ respectively and ϕ, λ for P_2 are $+37^\circ\, 37'\, 08''$ and $-122^\circ\, 22'\, 30''$ respectively.

Computed azimuths are $\alpha_{12} = 54^\circ\, 57'\, 06.932985''$ and $\alpha_{21} = 303^\circ\, 01'\, 14.140673''$, and

$s = 8\,246\,278.910557\,\text{m}$.

```
>> GEA_inverse

//////////////////////////////////////
// Great Elliptic Arc: Inverse Case //
//////////////////////////////////////

ellipsoid parameters
a   =  6378137.000000000
f   = 1/298.257223563000
b   =  6356752.314245179
e2  =  6.694379990141e-003
ep2 =  6.739496742276e-003
n   =  9.362215099752e-004

Great Elliptic parameters
a   =  6378137.000000000
f   = 1/534.561645318583
b   =  6366205.472446805
e2  =  3.737883789195e-003
ep2 =  3.751907985232e-003
n   =  9.362215099752e-004

Latitude  P1 =   35 45 55.000000 (D M S)
Longitude P1 =  140 23  8.000000 (D M S)

Latitude  P2 =   37 37  8.000000 (D M S)
Longitude P2 = -122 22 30.000000 (D M S)

Latitude   V =   48 26 49.347671 (D M S)
Longitude  V = -169 17 28.736205 (D M S)
Longitude  E =  100 42 31.263795 (D M S)

Cartesian coordinates
            X               Y               Z
P1   -3991399.691755  3303668.240372  3707090.313132
P2   -2708541.636331 -4272097.791174  3872024.259791
```

```
Constants of Great Elliptic Section
A   = 28628935911436.754000000
B   = -5413987974240.103500000
C   = 25999772788510.516000000

Azimuth of Great Elliptic section P1-P2
Az12  =  54 57  6.932985  (D M S)

Azimuth of Great Elliptic section P2-P1
Az21  = 303  1 14.140673  (D M S)

Great Elliptic Arc distances
M1 =  5702548.255829644
M2 =  6069943.563414550
Q  = 10009385.364900835
s = 2*Q-(M1+M2) =  8246278.910557477
s = M2-M1       =   367395.307584906


>>
```

TEST LINE: TOKYO → SAN FRANCISCO

Tokyo (NRT Airport):	$\phi = +\,35^\circ\,45'\,55''$	$\lambda = +\,140^\circ\,23'\,08''$
San Francisco (SFO Airport):	$\phi = +\,37^\circ\,37'\,08''$	$\lambda = -\,122^\circ\,22'\,30''$
WGS84 ellipsoid:	$a = 6378137$ m	$f = 1/298.257223563$

Point	Latitude	Longitude	Distance (m)	Forward Az	Reverse Az
Tokyo	35°45′55.0000″	140°23′08.0000″		54°57′06.9330″	
1	40°32′14.5095″	150°	994460.854		240°52′49.1340″
2	44°07′38.1588″	160°	1909191.293		247°36′09.3254″
3	46°32′28.3797″	170°	2737000.671		254°42′06.2015″
4	47°56′48.7303″	180°	3509459.054		262°01′50.4887″
5	48°26′41.5154″	-170°	4254408.310		269°28′16.4459″
Vertex	48°26′49.3477″	-169°17′28.7362″	4306837.109		270°
6	48°04′16.1310″	-160°	4997564.511		276°55′08.3743″
7	46°47′55.4369″	-150°	5764499.807		284°16′13.7972″
8	44°32′10.8186″	-140°	6582642.262		291°24′37.1081″
9	41°07′32.7972″	-130°	7482970.389		298°11′47.3337″
San Francisco	37°37′08.0000″	-122°37′08.0000″	8246278.910	54°57′06.9330″	303°01′14.1407″

Table 1: Great elliptic arc Tokyo→San Francisco and ten intermediate points (including the vertex). Distances are great elliptic arc lengths from Tokyo.

Line	Distance (m)	Forward Az	Reverse Az
Tokyo – San Francisco	8246278.910	54°57′06.9330″	303°01′14.1407″
Tokyo – 1	994460.854	54°57′06.9330″	240°52′49.1340″
1 – 2	914730.439	60°52′49.1340″	247°36′09.3254″
2 – 3	827809.378	67°36′09.3254″	254°42′06.2015″
3 – 4	772458.383	74°42′06.2015″	262°01′50.4887″
4 – 5	744949.256	82°01′50.4887″	269°28′16.4459″
5 – 6	743138.201	89°28′16.4459″	276°55′08.3743″
6 – 7	766935.296	96°55′08.3743″	284°16′13.7972″
7 – 8	818142.455	104°16′13.7972″	291°24′37.1081″
8 – 9	900328.127	111°24′37.1081″	298°11′47.3337″
9 – San Francisco	763308.521	118°11′47.3337″	303°01′14.1407″

Table 2: Distances and azimuths of sections of great elliptic arc Tokyo→San Francisco.

Geodesic Tokyo→San Francisco:

$\alpha_{12} = 54^\circ\, 49'\, 04.5523''$ $s = 8246271.872$ m $\alpha_{21} = 303^\circ\, 09'\, 21.9006''$

Geodesic − great elliptic arc = 7.038 m.

DIFFERENCE IN LENGTH BETWEEN A GEODESIC AND A GREAT ELLIPTIC ARC LENGTH

There are five curves of interest in geodesy; the geodesic, the normal section, the great elliptic arc the loxodrome and the curve of alignment.

The geodesic between P_1 and P_2 on an ellipsoid is the unique curve on the surface defining the shortest distance; all other curves will be longer in length. The normal section curve P_1P_2 is a plane curve created by the intersection of the normal section plane containing the normal at P_1 and also P_2 with the ellipsoid surface. And as we have shown (Deakin & Hunter 2010b) there is the other normal section curve P_2P_1. The curve of alignment (Deakin & Hunter 2010b, Thomas 1952) is the locus of all points P such that the normal section plane at P also contains the points P_1 and P_2. The curve of alignment is very close to a geodesic. The great elliptic arc is the plane curve created by intersecting the plane containing P_1, P_2 and the centre O with the surface of the ellipsoid and the loxodrome is the curve on the surface that cuts each meridian between P_1 and P_2 at a constant angle.

Approximate equations for the difference in length between the geodesic, the normal section curve and the curve of alignment were developed by Clarke (1880, p. 133) and

Bowring (1972, p. 283) developed an approximate equation for the difference between the geodesic and the great elliptic arc. Following Bowring (1972), let

$s =$ geodesic length
$L =$ normal section length
$D =$ great elliptic length
$S =$ curve of alignment length

then

$$\begin{aligned} L - s &= \frac{e^4}{90} s \left(\frac{s}{R} \right)^4 \cos^4 \phi_1 \sin^2 \alpha_{12} \cos^2 \alpha_{12} + \cdots \\ D - s &= \frac{e^4}{24} s \left(\frac{s}{R} \right)^2 \sin^2 \phi_1 \cos^2 \phi_1 \sin^2 \alpha_{12} + \cdots \\ S - s &= \frac{e^4}{360} s \left(\frac{s}{R} \right)^4 \cos^4 \phi_1 \sin^2 \alpha_{12} \cos^2 \alpha_{12} + \cdots \end{aligned} \tag{95}$$

where R can be taken as the radius of curvature in the prime vertical at P_1. Now for a given value of *s,* say the geodesic Tokyo to San Francisco on the WGS84 ellipsoid, $D - s$ will be a maximum if $\phi_1 = 45^\circ$ and $\alpha_{12} = 90^\circ$ in which case $\sin^2 \phi_1 \cos^2 \phi_1 \sin^2 \alpha_{12} = \frac{1}{4}$, thus

$$\left(D - s \right) < \frac{e^4}{96} s \left(\frac{s}{R} \right)^4 \tag{96}$$

For the WGS84 ellipsoid where $f = 1/298.257223563$, $e^2 = f\left(2 - f\right)$, and for $s = 8246271.872\text{ m}$ (8246.272 km) and $R = 6385442.306\text{ m}$, equation (96) gives $D - s < 10.707\text{ m}$. And from Table 1 $D = 8246278.910$ and $D - s = 7.038\text{ m}$ which satisfies inequality (96).

APPENDIX 1: Vectors

Vectors are very useful for describing various physical quantities or relationships such as force, velocity, acceleration, distance between objects, etc., that have both magnitude and direction. Vectors are represented by arrows between points or analytically by symbols such as $\overrightarrow{OP}$, or boldface characters **A** or **a**. The magnitude of a vector is denoted by $\left|\overrightarrow{OP}\right|, \left|\mathbf{A}\right|$ or $\left|\mathbf{a}\right|$ but it is also common to use A or a to represent the magnitude of vectors **A** or **a**.

A scalar, on the other hand, is a quantity having magnitude but no direction, e.g., mass, length, time, temperature and any real number.

Laws of Vector Algebra: If **A**, **B** and **C** are vectors and m and n are scalars then

1.	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Commutative law for Addition
2.	$\mathbf{A} + \left(\mathbf{B} + \mathbf{C}\right) = \left(\mathbf{A} + \mathbf{B}\right) + \mathbf{C}$	Associative law for Addition
3.	$m\mathbf{A} = \mathbf{A}m$	Commutative law for Multiplication
4.	$m\left(n\mathbf{A}\right) = \left(mn\right)\mathbf{A}$	Associative law for Multiplication
5.	$\left(m + n\right)\mathbf{A} = m\mathbf{A} + n\mathbf{A}$	Distributive law
6.	$m\left(\mathbf{A} + \mathbf{B}\right) = m\mathbf{A} + m\mathbf{B}$	Distributive law

A unit vector is a vector having unit magnitude (a magnitude of one). Unit vectors are denoted by $\hat{\mathbf{A}}$ or $\hat{\mathbf{a}}$ and

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{\left|\mathbf{A}\right|} = \frac{\mathbf{A}}{A}$$

Any vector **A** can be represented by a unit vector $\hat{\mathbf{A}}$ in the direction of **A** multiplied by the magnitude of A. That is, $\mathbf{A} = A\hat{\mathbf{A}}$

In an *x,y,z* Cartesian reference frame, the vector

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

has component vectors $A_1\mathbf{i}$, $A_2\mathbf{j}$ and $A_3\mathbf{k}$ in the *x, y* and *z* directions respectively, where **i**, **j** and **k** are unit vectors in the *x, y* and *z* directions. A_1, A_2 and A_3 are scalar components. The magnitude of **A** is

$$A = \left|\mathbf{A}\right| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

The unit vector of **A** is

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A} = \frac{A_1}{A}\mathbf{i} + \frac{A_2}{A}\mathbf{j} + \frac{A_3}{A}\mathbf{k}$$

The scalar product (or dot product) of two vectors $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ is defined as the product of the magnitudes of **A** and **B** multiplied by the cosine of the angle between them, or

$$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = AB\cos\theta$$

and $$\mathbf{A} \bullet \mathbf{B} = \left(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}\right) \bullet \left(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}\right) = A_1B_1 + A_2B_2 + A_3B_3$$

Note that $\mathbf{A} \bullet \mathbf{B}$ is a scalar and not a vector.

The following laws are valid for scalar products:

1. $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$ Commutative law
2. $\mathbf{A} \bullet \left(\mathbf{B} + \mathbf{C}\right) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$ Distributive law
3. $m\left(\mathbf{A} \bullet \mathbf{B}\right) = \left(m\mathbf{A}\right) \bullet \mathbf{B} = \left(\mathbf{A} \bullet \mathbf{B}\right)m$ where m is a scalar
4. $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1;\quad \mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$
5. $\mathbf{A} \bullet \mathbf{A} = a^2 = a_1^2 + a_2^2 + a_3^2$
6. If $\mathbf{A} \bullet \mathbf{B} = 0$, and **A** and **B** are not null vectors then **A** and **B** are perpendicular.

The vector product (or cross product) of two vectors $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ is a vector $\mathbf{P} = \mathbf{A} \times \mathbf{B}$ where **P** is a vector perpendicular to the plane containing **A** and **B**. The magnitude of **P** is defined as the product of the magnitudes of **A** and **B** multiplied by the sine of the angle between them. The vector product is often expressed as

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\ \hat{\mathbf{P}} = AB\sin\theta\,\hat{\mathbf{P}}$$

where $\hat{\mathbf{P}}$ is a perpendicular unit vector and the direction of **P** is given by the *right-hand-screw rule,* i.e., if **A** and **B** are in the plane of the head of a screw, then a clockwise rotation of **A** to **B** through an angle θ would mean that the direction of **P** would be the same as the direction of advance of a right-handed screw turned clockwise. The cross product can be written as the expansion of a determinant as

$$\mathbf{P} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} (+) & (-) & (+) \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \left(A_2B_3 - A_3B_2\right)\mathbf{i} - \left(A_1B_3 - A_3B_1\right)\mathbf{j} + \left(A_1B_2 - A_2B_1\right)\mathbf{k}$$

Note here that the mnemonics $(+)$, $(-)$, $(+)$ are an aid to the evaluation of the determinant. The perpendicular vector $\mathbf{P} = P_1\mathbf{i} + P_2\mathbf{j} + P_3\mathbf{k}$ has scalar components $P_1 = \left(A_2B_3 - A_3B_2\right)$, $P_2 = -\left(A_1B_3 - A_3B_1\right)$ and $P_3 = \left(A_1B_2 - A_2B_1\right)$.

The following laws are valid for vector products:

1. $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$ [Commutative law for cross products fails]
2. $\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$ Distributive law
3. $m\left(\mathbf{A}\times\mathbf{B}\right) = \left(m\mathbf{A}\right)\times\mathbf{B} = \mathbf{A}\times\left(m\mathbf{B}\right) = \left(\mathbf{A}\times\mathbf{B}\right)m$ where m is a scalar
4. $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$; $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$

Triple products

Scalar and vector multiplication of three vectors **A**, **B** and **C** may produce meaningful products of the form $\left(\mathbf{A}\bullet\mathbf{B}\right)\mathbf{C}$, $\mathbf{A}\bullet\left(\mathbf{B}\times\mathbf{C}\right)$ and $\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)$. The following laws are valid:

1. $\left(\mathbf{A}\bullet\mathbf{B}\right)\mathbf{C} \neq \mathbf{A}\left(\mathbf{B}\bullet\mathbf{C}\right)$
2. $\mathbf{A}\bullet\left(\mathbf{B}\times\mathbf{C}\right) = \mathbf{B}\bullet\left(\mathbf{C}\times\mathbf{A}\right) = \mathbf{C}\bullet\left(\mathbf{A}\times\mathbf{B}\right)$ (scalar triple products)
3. $\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right) \neq \left(\mathbf{A}\times\mathbf{B}\right)\times\mathbf{C}$
4. $\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right) = \left(\mathbf{A}\bullet\mathbf{C}\right)\mathbf{B} - \left(\mathbf{A}\bullet\mathbf{B}\right)\mathbf{C}$ (vector triple products)
   $\left(\mathbf{A}\times\mathbf{B}\right)\times\mathbf{C} = \left(\mathbf{A}\bullet\mathbf{C}\right)\mathbf{B} - \left(\mathbf{B}\bullet\mathbf{C}\right)\mathbf{A}$

Differentiation of vectors

If **A**, **B** and **C** are differentiable vector functions of a scalar u, and φ is a differentiable scalar function of u, then

1. $\dfrac{d}{du}\left(\mathbf{A}+\mathbf{B}\right) = \dfrac{d\mathbf{A}}{du} + \dfrac{d\mathbf{B}}{du}$
2. $\dfrac{d}{du}\left(\mathbf{A}\bullet\mathbf{B}\right) = \mathbf{A}\bullet\dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du}\bullet\mathbf{B}$
3. $\dfrac{d}{du}\left(\mathbf{A}\times\mathbf{B}\right) = \mathbf{A}\times\dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du}\times\mathbf{B}$
4. $\dfrac{d}{du}\left(\varphi\mathbf{A}\right) = \varphi\dfrac{d\mathbf{A}}{du} + \dfrac{d\varphi}{du}\mathbf{A}$
5. $\dfrac{d}{du}\left\{\mathbf{A}\bullet\left(\mathbf{B}\times\mathbf{C}\right)\right\} = \mathbf{A}\bullet\left(\mathbf{B}\times\dfrac{d\mathbf{C}}{du}\right) + \mathbf{A}\bullet\left(\dfrac{d\mathbf{B}}{du}\times\mathbf{C}\right) + \dfrac{d\mathbf{A}}{du}\bullet\left(\mathbf{B}\times\mathbf{C}\right)$
6. $\dfrac{d}{du}\left\{\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)\right\} = \mathbf{A}\times\left(\mathbf{B}\times\dfrac{d\mathbf{C}}{du}\right) + \mathbf{A}\times\left(\dfrac{d\mathbf{B}}{du}\times\mathbf{C}\right) + \dfrac{d\mathbf{A}}{du}\times\left(\mathbf{B}\times\mathbf{C}\right)$

Appendix 2: Taylor's theorem

This theorem, due to the English mathematician Brook Taylor (1685–1731) enables a function $f(x)$ near a point $x = a$ to be expressed from the values $f(a)$ and the successive derivatives of $f(x)$ evaluated at $x = a$.

Taylor's polynomial may be expressed in the following form

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \\ &\quad + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n \end{aligned} \tag{97}$$

where R_n is the remainder after n terms and $f'(a)$, $f''(a),\ldots$ etc. are derivatives of the function $f(x)$ evaluated at $x = a$.

Taylor's theorem can also be expressed as power series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{98}$$

where $f^{(k)}(a) \equiv \left[\frac{d^k}{dx^k} f(x)\right]_{x=a}$

The Taylor series for the function $\left(1 - e^2 \sin^2\phi\right)^{-\frac{3}{2}}$ evaluated at $e = 0$ is

$$\left(1 - e^2 \sin^2\phi\right)^{-\frac{3}{2}} = 1 + \frac{3}{2}e^2\sin^2\phi + \frac{15}{8}e^4\sin^4\phi + \frac{35}{16}e^6\sin^6\phi + \cdots$$

and the coefficients of even powers of e are binomial coefficients $\frac{3}{2}$, $\frac{3\cdot 5}{2\cdot 4}$, $\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}$, etc.

APPENDIX 3: Reversion of a series

If we have an expression for a variable z as a series of powers or functions of another variable y then we may, by a reversion of the series, find an expression for y as series of functions of z. Reversion of a series can be done using Lagrange's theorem, a proof of which can be found in Bromwich (1991).

Suppose that

$$y = z + xF(y) \quad \text{or} \quad z = y - xF(y) \tag{99}$$

then Lagrange's theorem states that for any f

$$\begin{aligned} f(y) = f(z) &+ \frac{x}{1!}F(z)f'(z) \\ &+ \frac{x^2}{2!}\frac{d}{dz}\left[\left\{F(z)\right\}^2 f'(z)\right] \\ &+ \frac{x^3}{3!}\frac{d^2}{dz^2}\left[\left\{F(z)\right\}^3 f'(z)\right] \\ &+ \cdots \\ &+ \frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\left\{F(z)\right\}^n f'(z)\right] \\ &+ \cdots \end{aligned} \tag{100}$$

As an example, consider the series for rectifying latitude μ

$$\mu = \phi + d_2 \sin 2\phi + d_4 \sin 4\phi + d_6 \sin 6\phi + \cdots \tag{101}$$

And we wish to find an expression for ϕ as a function of μ.

Comparing the variables in equations (101) and (99), $z = \mu$, $y = \phi$ and $x = -1$; and if we choose $f(y) = y$ then $f(z) = z$ and $f'(z) = 1$. So equation (101) can be expressed as

$$\mu = \phi + F(\phi) \tag{102}$$

and Lagrange's theorem gives

$$\begin{aligned} \phi = \mu - F(\mu) + \frac{1}{2}\frac{d}{d\mu}\left[\left\{F(\mu)\right\}^2\right] - \frac{1}{6}\frac{d^2}{d\mu^2}\left[\left\{F(\mu)\right\}^3\right] + \frac{1}{24}\frac{d^3}{d\mu^3}\left[\left\{F(\mu)\right\}^4\right] - \cdots \\ + \frac{(-1)^n}{n!}\frac{d^{n-1}}{d\mu^{n-1}}\left[\left\{F(\mu)\right\}^n\right] + \cdots \end{aligned} \tag{103}$$

where

$$F(\phi) = d_2 \sin 2\phi + d_4 \sin 4\phi + d_6 \sin 6\phi + \cdots$$

and so

$$F(\mu) = d_2 \sin 2\mu + d_4 \sin 4\mu + d_6 \sin 6\mu + \cdots$$

APPENDIX 4: *Maxima*

We now show the *Maxima* 'program' used in determining the coefficients of equations (13), (23) and (27) saved as a text file `Tseng_nSeries_Horner.mac` and run in batch mode silently.

```
/***************************************************************************
   Maxima program to develop series equations for:

    1. meridian distance M as a function of latitude B
       M = a/(1+n){c0*B + c2*sin(2B) + c4*sin(4B) + c6*sin(6B) + ...}

    2. rectifying latitude u as a function of latitude B
       u = B + d2*sin(2B) + d4*sin(4B) + d6*sin(6B) + ...

    3. latitude B as a function of rectifying latitude u
       B = u + D2*sin(2u) + D4*sin(4u) + D6*sin(6u) + ...

   The coefficients in each of these equations are functions of powers
   of the third flattening n = (a-b)/(a+b) where a and b (a>b) are the
   semi-axes of the reference ellipsoid.

   path and file name:
   D:\Projects\Geospatial\Geodesy\Great Elliptic Arc\Tseng
     \Tseng_nSeries_Horner.mac
****************************************************************************/

/* ------------------
   Lagrange reversion
   ------------------

   Suppose that we have a series expression for conformal latitude b as a
   function of latitude B having the form

   b = B + g2*sin(2*B) + g4*sin(4*B) + g6*sin(6*B) + ...      (1)

   g2, g4, g6, etc. are coefficients containing powers of eps.

   We wish to reverse the series (1) to obtain latitude B as a function
   of conformal latitude b having the form

   B = b + G2*sin(2*b) + G4*sin(4*b) + G6*sin(6*b) + ...      (2)

   G2, G4, G6, etc. are coefficients containing powers of eps.

   Write

   b = B + F(B)
   B = b + f(b)
   F(B) = sum(g[2*k]*sin(2*k*B)), k = 1 to inf
   F(b) = sum(g[2*k]*sin(2*k*b)), k = 1 to inf
   f(b) = sum(G[2*k]*sin(2*k*b)), k = 1 to inf

   Lagrange's inversion theorem gives

   f(b) = - F(b)
          + 1/2!*diff1[{F(b)}^2]
          - 1/3!*diff2[{F(b)}^3]
          + 1/4!*diff3[{F(b)}^4]
          - ...                             (3)
   or

   f(b) = sum((-1)^n/n! * diff(F(b)^n,b,n-1)), n = 1 to inf
```

```
   The following Maxima code for function Lagrange() was sent to me by
   Charles Karney 03-Jun-2010 and has been slightly modified by changing
   the function name and certain variable names.

       reverse
          var2 = expr(var1) = series in eps
       to
          var1 = revertexpr(var2) = series in eps
   Require that expr(var1) = var1 to order eps^0.  This throws in a
   trigreduce to convert to multiple angle trig functions.
*/

Lagrange(expr,var1,var2,eps,pow):=block
  ([b_acc:1,B_acc:0,dB],
   dB:ratdisrep(taylor(expr-var1,eps,0,pow)),
   dB:subst([var1=var2],dB),
   for n:1 thru pow do
       (b_acc:trigreduce(ratdisrep(taylor(-dB*b_acc/n,eps,0,pow))),
        B_acc:B_acc+expand(diff(b_acc,var2,n-1))),
   var2+B_acc)$

/*********************************************************
  FIRST: derive the series for meridian distance M as a
  function of the latitude B
*********************************************************/

/* the order to compute */
maxpow:5$

/* Integrand of the function for meridian distance */
Fintegrand:((1-n^2)^2*(1+n^2+2*n*cos(2*B))^(-3/2))$

/* expand the integrand into a Taylor series */
F:taylor(Fintegrand,n,0,maxpow)$

/* integrate the Taylor series w.r.t. latitude B */
f:integrate(F,B)$

/* reduce products and powers of sines and cosines to those of multiples */
f:trigreduce(f)$

/* expand the function */
f:expand(f)$

/* print equation for MERIDIAN DISTANCE */
print(" ");
print("equation for MERIDIAN DISTANCE M as a function of LATITUDE B");
print("M = a/(1+n){c0*B + c2*sin(2B) + c4*sin(4B) + c6*sin(6B) + ...}");
print(" ");

/* gather the coefficients of B and assign to c0 */
c0:coeff(f,B)$

/* print coefficients c0, c2, c4, c6, ... */
print("c0 = ",c0);
for i thru maxpow do
  print(c[2*i] = coeff(f,sin(2*i*B)));

/* print coefficients c0, c2, c4, c6, ... in Horner form */
print(" ");
print("coefficients in Horner form");
print(" ");
print("c0 = ",horner(c0,n));
for i thru maxpow do
  print(c[2*i] = horner(coeff(f,sin(2*i*B)),n));
```

```
/*********************************************************
  SECOND: derive the series for rectifying latitude u as a
  function of the latitude B
**********************************************************/

/* Divide all the coefficients of f by c0 which is the same
   as multiplying f by 1/c0 */

/* find a Taylor series for X = 1/c0 */
X:taylor(c0^-1,n,0,maxpow)$

/* multiply f with X and assign the result to u
   u will now contain the coefficients of the series
   for rectifying latitude */
u:f*X$

/* Expand the function u */
u:expand(u)$

/* group coefficients of u and sin(2*i*B) */
coeff(u,B)*B+sum(coeff(u,sin(2*i*B))*sin(2*i*B),i,1,maxpow)$

/* print equation for RECTIFYING LATITUDE */
print(" ");
print("equation for RECTIFYING LATITUDE u as a function of LATITUDE B");
print("u = B + d2*sin(2B) + d4*sin(4B) + d6*sin(6B) + ...");
print(" ");

/* print coefficients d2, d4, d6, ... */
for i thru maxpow do
  print(d[2*i] = coeff(u,sin(2*i*B)));

/* print coefficients d2, d4, d6, ... in Horner form */
print(" ");
print("coefficients in Horner form");
print(" ");
for i thru maxpow do
  print(d[2*i] = horner(coeff(u,sin(2*i*B)),n));

/**********************************************************
  THIRD: Reverse the series u = B + d2*sin(2B) + ...
  to give B = u + D2*sin(2B) + ...
***********************************************************/

/* copy u into uexpr and kill the earlier definition of u */
uexpr:u$
kill(u)$

/* reverse the series u = F(B,n) to give B = f(u,n) */
Bexpr:Lagrange(uexpr,B,u,n,maxpow)$

/* print equation for LATTUDE B */
print(" ");
print("equation for LATITUDE B as a function of RECTIFYING LATITUDE u");
print("B = u + D2*sin(2u) + D4*sin(4u) + D6*sin(6u) + ...");
print(" ");

/* print coefficients D2, D4, D6, ... */
for i thru maxpow do
  print(D[2*i] = coeff(Bexpr,sin(2*i*u)));

/* print coefficients D2, D4, D6, ... in Horner form */
print(" ");
print("coefficients in Horner form");
print(" ");
for i thru maxpow do
  print(D[2*i] = horner(coeff(Bexpr,sin(2*i*u)),n));
```

SERIES FOR GREAT ELLIPTIC ARC COMPUTATIONS:
Meridian distance M,
Rectifying latitude u,
Latitude B,
(normal and Horner form)

```
In the Maxima output that follows:
B is latitude
u is rectifying latitude
n is third flattening of the ellipsoid
a,b are semi-axes of reference ellipsoid (a>b)
```

OUTPUT FROM MAXIMA RUNNING “Tseng_nSeries_Horner.mac” IN BATCH MODE SILENTLY

```
coefficients to order n^5


Maxima 5.24.0 http://maxima.sourceforge.net
using Lisp GNU Common Lisp (GCL) GCL 2.6.8 (a.k.a. GCL)
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
(%i1)

equation for MERIDIAN DISTANCE M as a function of LATITUDE B
M = a/(1+n){c0*B + c2*sin(2B) + c4*sin(4B) + c6*sin(6B) + ...}

       4    2
      n    n
c0 =  -- + -- + 1
      64   4
        5      3
     3 n    3 n    3 n
c  = ---- + ---- - ---
 2   128     16     2
         2       4
     15 n    15 n
c  = ----- - -----
 4    16      64
          5       3
     175 n    35 n
c  = ------ - -----
 6    768      48
          4
     315 n
c  = ------
 8    512
             5
        693 n
c   = - ------
 10      1280
```

equation for MERIDIAN DISTANCE M as a function of LATITUDE B
M = a*(1-e^2){c0*B + c2*sin(2B) + c4*sin(4B) + c6*sin(6B) + ...}
coefficients in Horner form

$$c0 = \frac{n^2 (n^2 + 16) + 64}{64}$$

$$c_2 = \frac{n (n^2 (3 n^2 + 24) - 192)}{128}$$

$$c_4 = \frac{n^2 (60 - 15 n^2)}{64}$$

$$c_6 = \frac{n^3 (175 n^2 - 560)}{768}$$

$$c_8 = \frac{315 n^4}{512}$$

$$c_{10} = - \frac{693 n^5}{1280}$$

equation for RECTIFYING LATITUDE u as a function of LATITUDE B
u = B + d2*sin(2B) + d4*sin(4B) + d6*sin(6B) + ...

$$d_2 = - \frac{3 n^5}{32} + \frac{9 n^3}{16} - \frac{3 n}{2}$$

$$d_4 = \frac{15 n^2}{16} - \frac{15 n^4}{32}$$

$$d_6 = \frac{105 n^5}{256} - \frac{35 n^3}{48}$$

$$d_8 = \frac{315 n^4}{512}$$

$$d_{10} = - \frac{693 n^5}{1280}$$

equation for RECTIFYING LATITUDE u as a function of LATITUDE B

u = B + d2*sin(2B) + d4*sin(4B) + d6*sin(6B) + ...

coefficients in Horner form

$$d_2 = \frac{n\,(n^2\,(18 - 3\,n^2) - 48)}{32}$$

$$d_4 = \frac{n^2\,(30 - 15\,n^2)}{32}$$

$$d_6 = \frac{n^3\,(315\,n^2 - 560)}{768}$$

$$d_8 = \frac{315\,n^4}{512}$$

$$d_{10} = -\frac{693\,n^5}{1280}$$

equation for LATITUDE B as a function of RECTIFYING LATITUDE u

B = u + D2*sin(2u) + D4*sin(4u) + D6*sin(6u) + ...

$$D_2 = \frac{269\,n^5}{512} - \frac{27\,n^3}{32} + \frac{3\,n}{2}$$

$$D_4 = \frac{21\,n^2}{16} - \frac{55\,n^4}{32}$$

$$D_6 = \frac{151\,n^3}{96} - \frac{417\,n^5}{128}$$

$$D_8 = \frac{1097\,n^4}{512}$$

$$D_{10} = \frac{8011\,n^5}{2560}$$

```
equation for LATITUDE B as a function of RECTIFYING LATITUDE u
B = u + D2*sin(2u) + D4*sin(4u) + D6*sin(6u) + ...
coefficients in Horner form

         2       2
     n (n  (269 n  - 432) + 768)
D  = ---------------------------
 2              512
      2          2
     n  (42 - 55 n )
D  = ---------------
 4        32
      3             2
     n  (604 - 1251 n )
D  = -----------------
 6         384
          4
     1097 n
D  = -------
 8     512
           5
      8011 n
D   = -------
 10    2560
(%i2)
```

The batch file contains a function `Lagrange()` that performs the series reversion using Lagrange's theorem. This function was sent to me by Charles F.F. Karney (email 03-Jun-2010 at 12:39), the author of 'Transverse Mercator with an accuracy of a few nanometres' *Journal of Geodesy,* Vol. 85, pp. 475-485, published online: 09-Feb-2011.

Changing a few lines of the batch file allows the coefficients of the 'e-series' in equations (11), (21) and (25). The output is shown below.

SERIES FOR GREAT ELLIPTIC ARC COMPUTATIONS:
Meridian distance M,
Rectifying latitude u,
Latitude B,
(normal and Horner form)

```
In the Maxima output that follows:
B is latitude
u is rectifying latitude
e is the eccentricity of the reference ellipsoid
a,b are semi-axes of reference ellipsoid (a>b)
```

OUTPUT FROM MAXIMA RUNNING "Tseng_eSeries_Horner.mac" IN BATCH MODE SILENTLY

```
coefficients to order e^10

Maxima 5.24.0 http://maxima.sourceforge.net
using Lisp GNU Common Lisp (GCL) GCL 2.6.8 (a.k.a. GCL)
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
(%i1)
```

equation for MERIDIAN DISTANCE M as a function of LATITUDE B

M = a*(1-e^2){b0*B + b2*sin(2B) + b4*sin(4B) + b6*sin(6B) + ...}

$$b0 = \frac{43659\, e^{10}}{65536} + \frac{11025\, e^{8}}{16384} + \frac{175\, e^{6}}{256} + \frac{45\, e^{4}}{64} + \frac{3\, e^{2}}{4} + 1$$

$$b_2 = - \frac{72765\, e^{10}}{131072} - \frac{2205\, e^{8}}{4096} - \frac{525\, e^{6}}{1024} - \frac{15\, e^{4}}{32} - \frac{3\, e^{2}}{8}$$

$$b_4 = \frac{10395\, e^{10}}{65536} + \frac{2205\, e^{8}}{16384} + \frac{105\, e^{6}}{1024} + \frac{15\, e^{4}}{256}$$

$$b_6 = - \frac{10395\, e^{10}}{262144} - \frac{105\, e^{8}}{4096} - \frac{35\, e^{6}}{3072}$$

$$b_8 = \frac{3465\, e^{10}}{524288} + \frac{315\, e^{8}}{131072}$$

$$b_{10} = - \frac{693\, e^{10}}{1310720}$$

coefficients in Horner form

$$b0 = \frac{e^{2}\,(e^{2}\,(e^{2}\,(e^{2}\,(43659\, e^{2} + 44100) + 44800) + 46080) + 49152) + 65536}{65536}$$

$$b_2 = \frac{e^{2}\,(e^{2}\,(e^{2}\,(e^{2}\,(- 72765\, e^{2} - 70560) - 67200) - 61440) - 49152)}{131072}$$

$$b_4 = \frac{e^{4}\,(e^{2}\,(e^{2}\,(10395\, e^{2} + 8820) + 6720) + 3840)}{65536}$$

$$b_6 = \frac{e^{6}\,(e^{2}\,(- 31185\, e^{2} - 20160) - 8960)}{786432}$$

$$b_8 = \frac{e^{8}\,(3465\, e^{2} + 1260)}{524288}$$

$$b_{10} = - \frac{693\, e^{10}}{1310720}$$

equation for RECTIFYING LATITUDE u as a function of LATITUDE B

u = B + g2*sin(2B) + g4*sin(4B) + g6*sin(6B) + ...

$$g_2 = - \frac{1533\, e^{10}}{32768} - \frac{141\, e^{8}}{2048} - \frac{111\, e^{6}}{1024} - \frac{3\, e^{4}}{16} - \frac{3\, e^{2}}{8}$$

$$g_4 = \frac{165\, e^{10}}{4096} + \frac{405\, e^{8}}{8192} + \frac{15\, e^{6}}{256} + \frac{15\, e^{4}}{256}$$

$$g_6 = - \frac{4935\, e^{10}}{262144} - \frac{35\, e^{8}}{2048} - \frac{35\, e^{6}}{3072}$$

$$g_8 = \frac{315\, e^{10}}{65536} + \frac{315\, e^{8}}{131072}$$

$$g_{10} = - \frac{693\, e^{10}}{1310720}$$

coefficients in Horner form

$$g_2 = \frac{e^2\,(e^2\,(e^2\,(e^2\,(- 1533\, e^2 - 2256) - 3552) - 6144) - 12288)}{32768}$$

$$g_4 = \frac{e^4\,(e^2\,(e^2\,(330\, e^2 + 405) + 480) + 480)}{8192}$$

$$g_6 = \frac{e^6\,(e^2\,(- 14805\, e^2 - 13440) - 8960)}{786432}$$

$$g_8 = \frac{e^8\,(630\, e^2 + 315)}{131072}$$

$$g_{10} = - \frac{693\, e^{10}}{1310720}$$

equation for LATITUDE B as a function of RECTIFYING LATITUDE u
B = u + G2*sin(2u) + G4*sin(4u) + G6*sin(6u) + ...

$$G_2 = \frac{20861\, e^{10}}{524288} + \frac{255\, e^{8}}{4096} + \frac{213\, e^{6}}{2048} + \frac{3\, e^{4}}{16} + \frac{3\, e^{2}}{8}$$

$$G_4 = \frac{197\, e^{10}}{4096} + \frac{533\, e^{8}}{8192} + \frac{21\, e^{6}}{256} + \frac{21\, e^{4}}{256}$$

$$G_6 = \frac{5019\, e^{10}}{131072} + \frac{151\, e^{8}}{4096} + \frac{151\, e^{6}}{6144}$$

$$G_8 = \frac{1097\, e^{10}}{65536} + \frac{1097\, e^{8}}{131072}$$

$$G_{10} = \frac{8011\, e^{10}}{2621440}$$

coefficients in Horner form

$$G_2 = \frac{e^{2}\,(e^{2}\,(e^{2}\,(e^{2}\,(20861\, e^{2} + 32640) + 54528) + 98304) + 196608)}{524288}$$

$$G_4 = \frac{e^{4}\,(e^{2}\,(e^{2}\,(394\, e^{2} + 533) + 672) + 672)}{8192}$$

$$G_6 = \frac{e^{6}\,(e^{2}\,(15057\, e^{2} + 14496) + 9664)}{393216}$$

$$G_8 = \frac{e^{8}\,(2194\, e^{2} + 1097)}{131072}$$

$$G_{10} = \frac{8011\, e^{10}}{2621440}$$

(%i2)

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